

Non-chaotic dynamics in general-relativistic and scalar-tensor cosmology

V. Faraoni, M.N. Jensen and S.A. Theuerkauf

*Physics Department, Bishop's University
Lennoxville, Québec, Canada J1M 1Z7*

Abstract

In the context of scalar-tensor models of dark energy and inflation, the dynamics of vacuum scalar-tensor cosmology are analysed without specifying the coupling function or the scalar field potential. A conformal transformation to the Einstein frame is used and the dynamics of general relativity with a minimally coupled scalar field are derived for a generic potential. It is shown that the dynamics are non-chaotic, thus settling an existing debate.

Keywords: scalar-tensor cosmology, dark energy, inflation.

PACS: 98.80.-k, 04.90.+e, 04.50.+h

1 Introduction

Recent cosmological observations have established that the universe is very close to being spatially flat, corresponding to vanishing curvature index K in the Friedmann-Lemaitre-Robertson-Walker (“FLRW”) line element

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - Kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right] \quad (1.1)$$

in comoving coordinates (t, r, θ, φ) . We use units in which the speed of light and Newton’s constant assume the value unity, the metric signature is $-, +, +, +$, and the other notations follow those of Ref. [1]. If K were exactly zero, it would not be possible to establish it observationally, due to unavoidable experimental errors. Notwithstanding this, the cosmic microwave background experiments [2] measuring a total energy density ρ of the universe close to the critical density $\rho_c = \frac{3H^2}{8\pi G}$ (where $H \equiv \dot{a}/a$ is the Hubble parameter and an overdot denotes differentiation with respect to the comoving time t) can be regarded as a reassuring validation of the theoretical prediction that the universe was taken extremely close to the $K = 0$ state by inflation early in its history [3].

Another surprising discovery, obtained with the study of type Ia supernovae at high redshifts [4], is that the cosmic expansion is accelerated. In the context of Einstein’s gravity this fact is explained by postulating that the pressure P of the cosmic fluid satisfies $P < -\rho/3$. In fact, the best fit to the observational data requires even more exotic properties for the *dark energy*, the fluid that accounts for 70% of the energy density of the universe in Einstein’s theory; if $w \equiv P/\rho$ denotes the effective equation of state parameter of the dark energy, there is marginal evidence that $w < -1$ [5]. This range of values of the parameter w corresponds to a Hubble parameter that is increasing with time (*superacceleration*): $\dot{H} > 0$ according to the Friedmann equation

$$\dot{H} = -\frac{\kappa}{2}(P + \rho) , \quad (1.2)$$

where $\kappa \equiv 8\pi G$ and G is Newton’s constant. Most dark energy models are based on scalar fields and, if the universe really superaccelerates, models based on general relativity with a canonical, minimally coupled, scalar field ϕ are unviable. In fact, the energy density and pressure of such a scalar field are

$$\rho = \frac{\dot{\phi}^2}{2} + V(\phi) , \quad P = \frac{\dot{\phi}^2}{2} - V(\phi) , \quad (1.3)$$

where $V(\phi)$ is the scalar field potential, and eq. (1.2) reduces to $\dot{H} = -\kappa\dot{\phi}^2/2 \leq 0$ (the limiting situation $\dot{H} = 0$ describes de Sitter solutions). Furthermore, the best fit to the

observational data requires a dynamical form of dark energy with an equation of state parameter changing with redshift, which would definitely exclude the cosmological constant as an explanation for dark energy (such a model is anyway unappealing because of the cosmological constant problem and the cosmic coincidence problem [6, 7]). For this reason, alternative models in which the universe can superaccelerate have been considered, including phantom fields with negative kinetic energy [8], scalar fields coupled non-minimally to the curvature [9, 10], or alternative gravity theories. In this paper, we consider scalar-tensor extensions of general relativity. The theories in this class [11, 12] generalize Brans and Dicke’s [13] theory, exhibit many features in common with string/M theory [14], are the arena for extended [15] and hyperextended [16] inflationary scenarios, and are widely used in cosmology [17, 18]. Scalar-tensor gravity is used to model inflation in the early universe or dark energy in the present late time era of the universe. Both phantom cosmology and the theory of a non-minimally coupled scalar field, capable of producing the superacceleration phenomenon, can be seen as scalar-tensor theories.

Because of the wide use of scalar-tensor gravity, motivated by the belief that the latter may be more fundamental than Einstein’s gravity, one would like to have a clear and general picture of scalar-tensor cosmology. In general relativity the dynamics of a particular (dark energy or inflationary) scenario is usually determined once the potential is fixed; in scalar-tensor gravity one also has free coupling functions adding extra degrees of freedom. However, it would be desirable to understand the dynamics with as much generality as possible without fixing these functions. This is what we set out to do in this paper and, on the basis of the observational data and of theoretical prejudice, we restrict ourselves to considering a spatially flat ($K = 0$) universe. In particular, we derive conclusions about the dynamics based on general assumptions about the potential (e.g., monotonic, *etc.*). To the best of our knowledge, this classification has not been explored even in the relatively simple case of general relativity with a minimally coupled scalar field, in which the potential $V(\phi)$ is the only unknown function (e.g., in inflationary scenarios). Moreover, the geometry of the phase space is rarely discussed, even in scenarios based on general relativity with specific choices of the potential $V(\phi)$, for which phase space studies exist in the literature (see Refs. [19]–[22] for Brans-Dicke theory) — usually only projections of the phase space are considered. The role of chaos in cosmology has received much attention since the early work on anisotropic universes approaching an initial singularity (mixmaster universes) [23].

A particularly interesting aspect of the cosmological dynamics is the presence or absence of chaos in a $K = 0$ FLRW universe with a single scalar field; this has been the subject of debate. While the presence of chaos in the dynamics has been suggested on the basis of numerical studies and of a Painlevé analysis [24, 25], it has been stated explicitly that chaos can not occur in the case of a single, minimally or non-minimally coupled,

scalar field [26]. This has led to regarding inflation in the early universe as playing a new role: by taking the universe extremely close to a spatially flat one, primordial inflation eliminates the possibility of chaos from the dynamics. This statement is based on an analytic study of the phase space and its dimensional reduction to a two-dimensional surface [26, 27]. Since this dimensional reduction can be generalized to any scalar-tensor $K = 0$ cosmology with a single scalar field, it would appear that chaos is absent. However, the statement is based on the two-dimensional nature of the phase space but the known theorems (e.g., [28]) do not apply to a multi-sheeted, non-compact and non-connected phase space such as the one of scalar-tensor cosmology. Therefore, the problem of the presence or absence of chaos remains open. We show in the following that, while determining the dynamics for general potentials, one is also able to exclude the presence of chaos in the dynamics of scalar-tensor $K = 0$ cosmologies.

In section 2 we approach the problem of determining the dynamics in the context of (spatially flat) scalar-tensor cosmology in the Jordan frame, while in section 3 we reduce it to the equivalent problem in the context of Einstein's theory by means of a conformal transformation to the Einstein frame. In section 4 we discuss the dynamics of the scale factor and of the scalar field for general potentials in general relativity, and then the conformal transformation is used to map back the results to the Jordan frame. Section 5 contains a discussion and the conclusions.

2 Scalar-tensor cosmology

We consider the scalar-tensor class of theories [11] described by the general action

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[\phi R - \frac{\omega(\phi)}{\phi} g^{ab} \nabla_a \phi \nabla_b \phi - V(\phi) \right]. \quad (2.1)$$

We do not add to the action terms describing “ordinary” matter (as opposed to scalar field ϕ) because we are interested in situations in which the cosmic dynamics are dominated by the scalar field, e.g., during early inflation or during the late time dark energy era. The field equations are

$$\begin{aligned} R_{ab} - \frac{1}{2} g_{ab} R &= \frac{\omega(\phi)}{\phi^2} \left(\nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \nabla^c \phi \nabla_c \phi \right) \\ &+ \frac{1}{\phi} (\nabla_a \nabla_b \phi - g_{ab} \square \phi) - \frac{V(\phi)}{2\phi} g_{ab}, \end{aligned} \quad (2.2)$$

$$\square \phi = \frac{1}{2\omega(\phi) + 3} \left[\phi \frac{dV}{d\phi} - 2V(\phi) - \frac{d\omega}{d\phi} \nabla^c \phi \nabla_c \phi \right], \quad (2.3)$$

where ∇_a is the covariant derivative operator associated with the metric g_{ab} , and $\square \equiv g^{ab}\nabla_a\nabla_b$. In a $K = 0$ FLRW metric these equations assume the form

$$H^2 = -H \frac{\dot{\phi}}{\phi} + \frac{\omega(\phi)}{6} \left(\frac{\dot{\phi}}{\phi} \right)^2 + \frac{V(\phi)}{6\phi}, \quad (2.4)$$

$$\dot{H} = -\frac{\omega(\phi)}{2} \left(\frac{\dot{\phi}}{\phi} \right)^2 + 2H \left(\frac{\dot{\phi}}{\phi} \right) + \frac{1}{2\phi(2\omega(\phi) + 3)} \left[\phi V' - 2V + \omega' \left(\frac{\dot{\phi}}{\phi} \right)^2 \right], \quad (2.5)$$

$$\ddot{\phi} + \left(3H + \frac{\dot{\omega}}{2\omega + 3} \right) \dot{\phi} = \frac{1}{2\omega + 3} (2V - \phi V'), \quad (2.6)$$

where a prime denotes differentiation with respect to ϕ . Eqs. (2.4)-(2.6) are respectively, the Hamiltonian constraint $H^2 = \kappa\rho_\phi/3$, $\dot{H} = -\kappa(\rho_\phi + P_\phi)/2$, and the Klein-Gordon equation for ϕ , where ρ_ϕ and P_ϕ are the effective energy density and pressure of the scalar. Only two equations in the set (2.4)-(2.6) are independent. We choose H and ϕ as dynamical variables (other works in the literature choose different variables for ease of manipulation in special cases, but the physical meaning of their results is somewhat obscured). To analyze the phase space it is convenient to regard the Hamiltonian constraint (2.4) as the algebraic equation for $\dot{\phi}$

$$\omega\dot{\phi}^2 - 6H\phi\dot{\phi} + (\phi V - 6H^2\phi^2) = 0 \quad (2.7)$$

with roots

$$\dot{\phi}_\pm(H, \phi) = \frac{1}{\omega(\phi)} \left(3H\phi \pm \sqrt{3H^2\phi^2(2\omega + 3) - \omega\phi V} \right). \quad (2.8)$$

Unless $\omega = 0$, the phase space is a two-dimensional surface $\Sigma = \Sigma^+ \cup \Sigma^-$ composed of two sheets Σ^\pm corresponding to the positive or negative sign in eq. (2.8), embedded in the three-dimensional space $(H, \phi, \dot{\phi})$. In actual fact, only the region $\phi > 0$ is of physical interest because it corresponds to positive gravitational coupling, and there are physical constraints on the variability of the effective gravitational coupling

$$G_{eff}(\phi) = \frac{2(\omega + 2)}{(2\omega + 3)\phi} \quad (2.9)$$

(see, e.g., Refs. [17, 18]).

In general, there is a dynamically forbidden region

$$\mathcal{F} = \{ (H, \phi) : 3H^2\phi^2[2\omega(\phi) + 3] - \omega(\phi)\phi V(\phi) < 0 \}, \quad (2.10)$$

corresponding to a negative argument of the square root in eq. (2.8). This forbidden region may be absent for specific choices of $\omega(\phi)$ and $V(\phi)$, but is present in most scenarios considered in the literature (see Ref. [29] for further details). The boundary of the forbidden region is composed of the only points where the two sheets touch each other and $\dot{\phi}$ is single valued, i.e., the set

$$\mathcal{B} \equiv \left\{ (H, \phi) : \dot{\phi} = \frac{3H\phi}{\omega(\phi)} \right\} = \Sigma^+ \cap \Sigma^- , \quad (2.11)$$

or

$$(H, \phi) : 3H^2\phi^2(2\omega + 3) = \phi\omega V \quad (2.12)$$

on \mathcal{B} , which defines implicitly a curve $H(\phi)$ in the space $(H, \phi, \dot{\phi})$.

Having chosen H and ϕ as dynamical variables, the equilibrium points of the system are necessarily de Sitter spaces with constant scalar field (H_0, ϕ_0) . If they exist, these can lie anywhere in the phase space $\Sigma \cap \{\dot{\phi} = 0\}$. According to the dynamical equations (2.4)-(2.6), there are two necessary and sufficient conditions for the existence of such de Sitter fixed points.

$$H_0^2 = \frac{V(\phi_0)}{6\phi_0} , \quad (2.13)$$

$$\phi_0 V'_0 - 2V_0 = 0 . \quad (2.14)$$

This two-dimensional structure of the phase space was reported earlier in the literature for minimally [30] and non-minimally coupled [31, 27] scalar fields. In Ref. [26] it was argued that the reduction of the phase space to two dimensions implies that chaos is impossible, and therefore that primordial inflation taking the universe extremely close to an exactly spatially flat FLRW space in addition to solving the problems of standard Big Bang cosmology [3], has the effect of inhibiting chaos. However, this conclusion is based on the Poincaré-Bendixson theory which applies to a two-dimensional phase space that is flat instead of curved, and to regions that are compact and connected. This is certainly not the case of the Σ phase space which, in addition to being double-sheeted and curved, extends to infinity in all directions and in general is not connected due to the presence of the forbidden region, which can consist of two or more separate “holes” in Σ (see Ref. [27] for examples).

At best, the argument of Ref. [26] arguing against the presence of chaos for the special case of nonminimally coupled fields can be applied (and generalized to arbitrary scalar-tensor cosmologies described by eqs. (2.4)-(2.6)) to compact regions of the phase space Σ

as follows. Consider a compact region \mathcal{C} in one of the sheets Σ^+ or Σ^- , lying away from the boundary \mathcal{B} of the forbidden region. Let (u, v) be smooth local coordinates covering \mathcal{C} (if u and v do not cover the entire region \mathcal{C} one can consider an atlas composed of two or more charts);

$$u = u(H, \phi) , \quad (2.15)$$

$$v = v(H, \phi) , \quad (2.16)$$

is a smooth map from a region of \mathbb{R}^2 to the sheet Σ^\pm , which is locally flat. Then, using the comoving time t as a parameter,

$$\begin{aligned} \dot{u} = & \frac{\partial u}{\partial H} \dot{H} + \frac{\partial u}{\partial \phi} \dot{\phi} = \left[-\frac{\omega}{2\phi^2} + \frac{\omega'}{2\phi(2\omega+3)} \right] \dot{\phi}^2 \frac{\partial u}{\partial H} + \left(\frac{2H}{\phi} \frac{\partial u}{\partial H} + \frac{\partial u}{\partial \phi} \right) \dot{\phi} \\ & + \frac{\phi V' - 2V}{2\phi(2\omega+3)} \frac{\partial u}{\partial H} , \end{aligned} \quad (2.17)$$

$$\begin{aligned} \dot{v} = & \frac{\partial v}{\partial H} \dot{H} + \frac{\partial v}{\partial \phi} \dot{\phi} = \left[-\frac{\omega}{2\phi^2} + \frac{\omega'}{2\phi(2\omega+3)} \right] \dot{\phi}^2 \frac{\partial v}{\partial H} + \left(\frac{2H}{\phi} \frac{\partial v}{\partial H} + \frac{\partial v}{\partial \phi} \right) \dot{\phi} \\ & + \frac{\phi V' - 2V}{2\phi(2\omega+3)} \frac{\partial v}{\partial H} . \end{aligned} \quad (2.18)$$

By using the expression (2.8) of $\dot{\phi}$ and the fact that u, v , and their derivatives only depend on H and ϕ , one can write eqs. (2.17) and (2.18) in the form of the autonomous system of first order ordinary differential equations for $(u(t), v(t))$

$$\dot{u} = f(u, v) , \quad (2.19)$$

$$\dot{v} = g(u, v) , \quad (2.20)$$

where

$$\begin{aligned} f(u, v) = & \left[-\frac{\omega}{2\phi^2} + \frac{\omega'}{2\phi(2\omega+3)} \right] \left[\frac{3H\phi \pm \sqrt{3H^2\phi^2(2\omega+3) - \omega\phi V}}{\omega} \right]^2 \frac{\partial u}{\partial H} \\ & + \frac{3H\phi \pm \sqrt{3H^2\phi^2(2\omega+3) - \omega\phi V}}{\omega} \left(\frac{2H}{\phi} \frac{\partial u}{\partial H} + \frac{\partial u}{\partial \phi} \right) \\ & + \frac{\phi V' - 2V}{2\phi(2\omega+3)} \frac{\partial u}{\partial H} , \end{aligned} \quad (2.21)$$

and

$$\begin{aligned}
g(u, v) &= \left[-\frac{\omega}{2\phi^2} + \frac{\omega'}{2\phi(2\omega+3)} \right] \left[\frac{3H\phi \pm \sqrt{3H^2\phi^2(2\omega+3) - \omega\phi V}}{\omega} \right]^2 \frac{\partial v}{\partial H} \\
&+ \frac{3H\phi \pm \sqrt{3H^2\phi^2(2\omega+3) - \omega\phi V}}{\omega} \left(\frac{2H}{\phi} \frac{\partial v}{\partial H} + \frac{\partial v}{\partial \phi} \right)
\end{aligned} \tag{2.22}$$

$$+ \frac{\phi V' - 2V}{2\phi(2\omega+3)} \frac{\partial v}{\partial H} . \tag{2.23}$$

The reduction of the dynamical system to a first order autonomous system with phase space consisting of a plane allows one to apply the standard Poincaré-Bendixson theory which guarantees that compact and connected regions of Σ^\pm (corresponding to compact and connected regions of \mathbb{R}^2) are free of chaos. However, this argument only proves the statement of Ref. [26] for such compact regions but not for the entire phase space Σ . This larger phase space can be studied in the context of a more comprehensive analysis of the dynamics that we propose in the next two sections.

3 Conformal mapping of the phase space

It is well known that the Jordan frame action (2.1) of scalar-tensor gravity can be mapped to the Einstein Hilbert action by means of the conformal transformation

$$g_{ab} \longrightarrow \tilde{g}_{ab} = \Omega^2 g_{ab}, \quad \Omega = \sqrt{\phi}, \tag{3.1}$$

and the scalar field redefinition $\phi \longrightarrow \tilde{\phi}(\phi)$ given by

$$d\tilde{\phi} = \sqrt{\frac{2\omega(\phi) + 3}{16\pi}} \frac{d\phi}{\phi} \tag{3.2}$$

(see Refs. [32, 33, 17] for reviews). In terms of conformally rescaled quantities in the Einstein frame, which are denoted by a tilde, the action (2.1) assumes the form

$$S = \int d^4x \sqrt{-g} \left[\frac{\tilde{R}}{16\pi G} - \frac{1}{2} \tilde{g}^{ab} \tilde{\nabla}_a \tilde{\phi} \tilde{\nabla}_b \tilde{\phi} - U(\tilde{\phi}) \right], \tag{3.3}$$

where

$$U(\tilde{\phi}) = \frac{V[\phi(\tilde{\phi})]}{\phi^2}. \tag{3.4}$$

The sign of the potential is preserved by the conformal transformation. By studying the dynamics in the Einstein frame and mapping back to the Jordan frame, one can study the dynamics and, in particular, the absence or presence of chaos.

Before we proceed, let us study the conformal cousins in the Einstein frame of the de Sitter fixed points living in the Jordan frame. It is easily concluded that de Sitter equilibrium points in the Jordan frame are mapped into de Sitter equilibrium points in the Einstein frame, and vice-versa. In fact, by using eq. (3.2) and the usual range of values of the coupling function $2\omega(\phi) + 3 > 0$ given by Solar System experiments and guaranteeing a positive-definite kinetic energy term for the scalar, one has $d\tilde{\phi}/d\tilde{t} = 0$ if and only if $d\phi/dt = 0$. In addition, eqs. (1.1) and (3.1) yield

$$d\tilde{s}^2 = \Omega^2 ds^2 = \Omega^2 (-dt^2 + a^2 d\vec{x}^2) = -d\tilde{t}^2 + \tilde{a}^2 d\vec{x}^2, \quad (3.5)$$

where $d\tilde{t} \equiv \sqrt{\phi} dt$ and $\tilde{a} \equiv \sqrt{\phi} a$. Hence,

$$\frac{d\tilde{\phi}}{d\tilde{t}} = \sqrt{\frac{2\omega + 3}{16\pi}} \phi^{-3/2} \frac{d\phi}{dt}. \quad (3.6)$$

Because we consider positive gravitational coupling $\phi > 0$, the sign of $d\tilde{\phi}/d\tilde{t}$ is the same as that of $\dot{\phi}$, and $d\phi/dt = 0$ if and only if $d\tilde{\phi}/d\tilde{t} = 0$, while

$$\frac{d\tilde{H}}{d\tilde{t}} = \frac{1}{\phi} \left(\frac{\ddot{\phi}}{2\phi} - \frac{3\dot{\phi}^2}{4\phi^{3/2}} + \dot{H} - \frac{H\dot{\phi}}{2\phi} \right). \quad (3.7)$$

Therefore, a de Sitter fixed point with constant scalar field $(\dot{H}, \dot{\phi}) = (0, 0)$ in the Jordan frame corresponds to a de Sitter fixed point with constant scalar field $(d\tilde{H}/d\tilde{t}, d\tilde{\phi}/d\tilde{t}) = (0, 0)$ in the Einstein frame (this fact is related to the scale invariance property of the exponential function). It is not true, as is instead stated in Ref. [34] that the conformal transformation becomes singular at the equilibrium points.

The condition (2.14) for the existence of the Jordan frame fixed point (H_0, ϕ_0) translates into the corresponding condition $dU/d\phi = 0$. In fact,

$$\frac{dU}{d\tilde{\phi}} = \frac{dU}{d\phi} \frac{d\phi}{d\tilde{\phi}} = \sqrt{\frac{16\pi}{2\omega(\phi) + 3}} \frac{1}{\phi} \left[V'(\phi) - \frac{2V(\phi)}{\phi} \right] \quad (3.8)$$

and eq. (2.14) implies $\left. \frac{dU}{d\phi} \right|_{\tilde{\phi}_0} = 0$, whereas $\phi = \phi_0 = \text{constant}$ is equivalent to $\tilde{\phi} = \tilde{\phi}_0 = \text{constant}$. Moreover, stability of the de Sitter fixed point in the Jordan frame

corresponds to stability of the corresponding de Sitter point in the Einstein frame, as shown in the following. Since

$$\tilde{H} \equiv \frac{1}{\tilde{a}} \frac{d\tilde{a}}{d\tilde{t}} = \frac{\dot{\phi}}{2\phi^{3/2}} + \frac{H}{\sqrt{\phi}} , \quad (3.9)$$

perturbations δH and $\delta\phi$ in the Jordan frame correspond to the Einstein frame perturbations

$$\delta\tilde{H} = \frac{\delta\dot{\phi}}{2\phi^{3/2}} + \frac{\delta H}{\sqrt{\phi_0}} - \frac{H_0}{2\phi_0^{3/2}} \delta\phi , \quad (3.10)$$

$$\delta\tilde{\phi} = \sqrt{\frac{2\omega(\phi_0) + 3}{16\pi}} \frac{\delta\phi}{\phi_0} , \quad (3.11)$$

to first order. If the Jordan frame fixed point (H_0, ϕ_0) is stable, perturbations δH and $\delta\phi$ do not grow (or decay rapidly) and the same can be concluded for the Einstein frame perturbations $\delta\tilde{H}, \delta\tilde{\phi}$; and vice-versa.

4 Dynamics in the conformally rescaled world

We finally proceed to study the dynamics in general relativity in the conformally rescaled world. For economy of notations we drop the tilde and in this section an overdot denotes differentiation with respect to the Einstein frame comoving time. The Einstein-Friedmann equations are

$$H^2 = \frac{\kappa}{6} \left[\dot{\phi}^2 + 2U(\phi) \right] , \quad (4.1)$$

$$\dot{H} = -\frac{\kappa}{2} \dot{\phi}^2 , \quad (4.2)$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dU}{d\phi} = 0 . \quad (4.3)$$

Since general relativity is a trivial case of scalar-tensor gravity, the structure of the phase space discussed in section 2 is still present, albeit some simplifications occur. The Hamiltonian constraint (4.1) can be seen as an algebraic equation for $\dot{\phi}$ with roots

$$\dot{\phi}_{\pm}(H, \phi) = \pm \sqrt{2 \left[\frac{3H^2}{\kappa} - U(\phi) \right]} , \quad (4.4)$$

which makes evident the double-sheeted structure of the two-dimensional phase space $\Sigma = \Sigma^+ \cup \Sigma^-$ embedded in the three-dimensional space $(H, \phi, \dot{\phi})$. In general there is a forbidden region

$$\mathcal{F} \equiv \{(H, \phi) : \frac{3H^2}{\kappa} < U(\phi)\} \quad (4.5)$$

with boundary

$$\mathcal{B} \equiv \Sigma^+ \cap \Sigma^- = \{(H, \phi, \dot{\phi}) : \frac{3H^2}{\kappa} = U(\phi), \dot{\phi} = 0\}. \quad (4.6)$$

The de Sitter equilibrium points (H_0, ϕ_0) are the only de Sitter spaces, while in general scalar-tensor cosmology de Sitter spaces with non-constant scalar field may be solutions, although they are not fixed points (cf. eq. (2.5)). Moreover, contrary to the case of section 2, all the de Sitter fixed points are forced to lie on the boundary $\mathcal{B} = \Sigma^+ \cap \Sigma^-$ between the upper and lower sheet, corresponding to $\dot{\phi} = 0$. Eqs. (4.1)-(4.3) yield the two conditions for the existence of de Sitter spaces (H_0, ϕ_0) (there are only two conditions because only two of the eqs. (4.1)-(4.3) are independent),

$$H_0^2 = \frac{\kappa}{3} U_0, \quad (4.7)$$

$$U'_0 = 0, \quad (4.8)$$

where $U_0 \equiv U(\phi_0)$ and $U'_0 \equiv \frac{dU}{d\phi}|_{\phi_0}$. de Sitter fixed points exist if the potential $U(\phi)$ has maxima, minima, or inflection points at $\phi_1, \phi_2, \dots, \phi_n, \dots$ with $U'(\phi_1) = U'(\phi_2) = \dots = U'(\phi_n) = \dots = 0$. One expects that if U has a maximum (respectively, a minimum) in ϕ_i , the corresponding fixed point $(\sqrt{\kappa U(\phi_i)/3}, \phi_i)$ will be unstable (respectively, stable) as we prove later.

Eq. (4.2) allows one to conclude immediately that there are no limit cycles (periodic orbits) because H is always decreasing (apart from the fixed point solutions) and can not come back to a previous value during the evolution of the system. Each point of \mathcal{B} where $U' \neq 0$ can be crossed by a trajectory only once, and in a definite direction specified by the sign of U' at that point.

In the upper sheet with boundary removed $\Sigma^+ \setminus \mathcal{B}$, it is $\dot{\phi} > 0$ and $\dot{H} < 0$, while in $\Sigma^- \setminus \mathcal{B}$ it is $\dot{\phi} < 0$ and $\dot{H} < 0$, hence $\dot{\phi}$ can not change sign away from the boundary \mathcal{B} . On $\mathcal{B} = \Sigma^+ \cap \Sigma^-$ it is $\dot{H} = 0, \dot{\phi} = 0, \ddot{\phi} = -U(\phi)$, as follows from the dynamical equations. The tangent to the orbits of the solutions parametrized by the time t in the space $(H, \phi, \dot{\phi})$ is the vector

$$\vec{T}(t) = (\dot{H}(t), \dot{\phi}(t), \ddot{\phi}(t)) \quad (4.9)$$

and, on \mathcal{B} ,

$$\vec{T}(t)|_{\mathcal{B}} = (0, 0, -U') . \quad (4.10)$$

On the boundary \mathcal{B} , the tangent \vec{T} can only be vertical and no motion along the H or ϕ directions can occur. This means that there can be no motion along the curve \mathcal{B} and portions of \mathcal{B} can not be parts of orbits of solutions. At points of \mathcal{B} where $U' = 0$, it is $\|\vec{T}\| = 0$ and there is no motion: these points are de Sitter fixed points. At points of \mathcal{B} where $U' > 0$, the tangent \vec{T} points downward along the negative $\dot{\phi}$ -axis and an orbit can only go from the upper sheet Σ^+ to the lower sheet Σ^- by crossing the boundary \mathcal{B} . If instead $U' < 0$ at points of \mathcal{B} , the tangent \vec{T} to the orbit is pointing upward in the positive $\dot{\phi}$ -direction and the orbit crosses from Σ^- to Σ^+ . This excludes the possibility that an orbit “bounces” on the boundary \mathcal{B} back to the sheet it came from. The possibility is not excluded that, in certain potentials, once the orbit has crossed, it changes component of the vertical velocity $\ddot{\phi}$ and comes back to the boundary to change sheet again. From eqs. (4.1)-(4.3) it follows that

$$\dot{H} + 3H^2 = \kappa U . \quad (4.11)$$

Since \dot{H} always decreases monotonically (except, of course, at the fixed points) either H tends to a finite limit (horizontal asymptote) H_0 as $t \rightarrow +\infty$, or else $H \rightarrow -\infty$. (For a given potential $U(\phi)$, there may be orbits going to a finite H_0 and other orbits going to $-\infty$, depending on the initial conditions).

If $H \rightarrow H_0$ as $t \rightarrow +\infty$, then $\dot{H} \rightarrow 0$ and eq. (4.2) implies that $\dot{\phi} \rightarrow 0$, which implies that also the scalar field ϕ approaches a horizontal asymptote ϕ_0 . In this situation, eq. (4.11) yields $3H_0^2 = \kappa U(\phi_0)$, i.e., we have a de Sitter fixed point satisfying eq. (4.7), which can only exist if $U'(\phi_0) = 0$. If $U' \neq 0$ for all values of ϕ (i.e., if $U(\phi)$ is strictly monotonic), there is the possibility that $H \rightarrow -\infty$ instead, which corresponds to a Big Crunch in a finite time, but there is also the possibility that $H(t) \rightarrow H_0$ asymptotically while $\phi(t) \rightarrow \pm\infty$ and both $\dot{H}, \dot{\phi} \rightarrow 0$ as $t \rightarrow +\infty$. An example is the exponential potential

$$U(\phi) = A \exp \left[\pm \sqrt{\frac{16\pi}{p}} \phi \right] \quad (p > 1) , \quad (4.12)$$

which gives power-law inflation [3]

$$a(t) = a_0 t^p , \quad (4.13)$$

$$\phi(t) = \sqrt{\frac{p}{4\pi}} \ln \left(\sqrt{\frac{8\pi A}{p(3p-1)}} t \right) , \quad (4.14)$$

for which $H = p/t \rightarrow 0$ while $\phi \rightarrow +\infty$ and $\dot{\phi} \rightarrow 0$.

We can now consider the situation in which $H \rightarrow -\infty$ and consider times sufficiently large so that $H < 0$. To proceed we need to make some assumptions on the potential: we first assume that U is *monotonic*, so U' has definite sign. The first possibility is $U' < 0$ for all ϕ ; then the Klein-Gordon equation $\ddot{\phi} = -3H\dot{\phi} - U'$ implies that in the upper sheet Σ^+ , where $\dot{\phi} > 0$, it is $\ddot{\phi} > 0$ and therefore $\phi \rightarrow +\infty$; the point (H, ϕ) representing the universe in the phase space goes to infinity without oscillating. In the lower sheet Σ^- , where $\dot{\phi} < 0$, either ϕ has a horizontal asymptote with $\phi \rightarrow \phi_0$, or else $\phi \rightarrow -\infty$. The first possibility is excluded because it would imply that $\dot{\phi} \rightarrow 0$ and, according to eq. (4.2), also $\dot{H} \rightarrow 0$, which contradicts the assumption that $H \rightarrow -\infty$; hence also in this case (H, ϕ) goes to infinity without oscillations.

In the case $U' > 0$ for all values of ϕ , take points in the upper sheet Σ^+ with $\dot{\phi} > 0$: then either $\phi(t) \rightarrow \phi_0$ (horizontal asymptote) or else $\phi(t) \rightarrow +\infty$. If ϕ has a horizontal asymptote ϕ_0 , i.e., if $\phi \rightarrow \phi_0$, then $\dot{\phi}(t) \rightarrow 0$ and also $\dot{H}(t) \rightarrow 0$: but this contradicts the assumption that $H(t) \rightarrow -\infty$. Then it must be $\phi(t) \rightarrow +\infty$ and the point (H, ϕ) goes to infinity without oscillations. If instead we pick a point in the lower sheet Σ^- with $\dot{\phi} < 0$, then $\ddot{\phi} = -3H\dot{\phi} - U' < 0$ and $\phi(t)$ goes to minus infinity without oscillations.

We can then consider the trivial case in which $U' = 0$ for all values of ϕ : this represents a cosmological constant and gives unviable early universe models in which inflation does not stop or dark energy models that are disfavoured (a scalar field is introduced to get away from the cosmological constant and its problems [7]); therefore, this is a case of mathematical interest that we include for completeness. In this case $\ddot{\phi} = -3H\dot{\phi}$, which integrates to $\dot{\phi} = C/a^3$ where C is a non-zero constant, and $\dot{\phi} \rightarrow \infty$: the point (H, ϕ) tends to infinity without oscillations.

Generally, we can consider a non-monotonic potential with a minimum, or possibly several maxima and minima. In this case the sign of $U(\phi)$ changes as ϕ evolves; this situation is best studied with a Ljapunov function.

First, assume that $U(\phi)$ has a single absolute minimum $U_0 \equiv U(\phi_0)$ attained at the single value ϕ_0 of the scalar field (examples are the widely used potentials $U = m^2\phi^2/2$ or $U = \lambda\phi^4$), then $(H_0, \phi_0) = (\sqrt{\kappa U_0/3}, \phi_0)$ is a fixed point. The function

$$L(H, \phi) = \frac{\dot{\phi}^2}{2} + U(\phi) - U_0 \quad (4.15)$$

is a Ljapunov function. In fact, $U(\phi) > 0 \ \forall \phi \neq \phi_0$ and $L(H, \phi) > 0 \ \forall (H, \phi) \neq (H_0, \phi_0)$. Moreover, $L(H_0, \phi_0) = 0$ since at the fixed point $\dot{\phi} = 0$. Along the orbits of the solutions we obtain, upon use of the Klein-Gordon equation,

$$\frac{dL}{dt} = \dot{\phi} (\ddot{\phi} + U') = -3H\dot{\phi}^2 < 0 \quad (4.16)$$

when $H > 0$, i.e., for all expanding universes. This guarantees that (H_0, ϕ_0) is an attractor with an attraction basin at least as wide as the $H > 0$ region of Σ .

The Ljapunov function $L(H, \phi)$ also allows us to immediately conclude that contracting de Sitter spaces with $H_0 = -\sqrt{\kappa U_0/3}$ are unstable because $H < 0$ and $\dot{L} > 0$ in a neighbourhood of this equilibrium point (it is well known that contracting de Sitter spaces are unstable — see, e.g., Ref. [35]).

If $U(\phi)$ has an absolute minimum that is assumed at two (or more) different values of ϕ , say $\phi_1, \phi_2, \dots, \phi_n, \dots$ then $U(\phi) > U(\phi_1) = U(\phi_2) = \dots = U(\phi_n) = \dots \equiv U_{min} \quad \forall \phi \notin \{\phi_1, \phi_2, \dots, \phi_n, \dots\}$ and $U(\phi)$ must assume maxima between the minima. In this case the phase space Σ will contain the attraction basins of the stable fixed points corresponding to the minima of $U(\phi)$ (the maxima corresponding to unstable fixed points), with a separatrix going through a fixed point between two adjacent attraction basins [34].

5 Discussion and conclusions

Based on the reported marginal observational evidence for present superacceleration of the universe, which can not be explained by general relativity with a canonical scalar field, we consider spatially flat homogeneous and isotropic cosmologies in scalar-tensor gravity. The dynamics are conveniently studied by a conformal mapping of the Jordan frame scalar-tensor theory into the Einstein frame, in which gravity reduces to general relativity and the rescaled scalar field is minimally coupled to gravity (a non-minimal coupling of the scalar to ordinary matter is irrelevant here because we consider the scalar to be the dominant form of “matter” in the universe, and other material sources can be neglected). Apart from the original motivation to study scalar-tensor gravity by using the mathematical trick of the conformal transformation to the Einstein frame, the study of the phase space and of the dynamics of general relativity with a minimally coupled scalar field is interesting in itself. We emphasize that this work is limited to spatially flat ($K = 0$) FLRW models because they describe our universe according to the recent observations of the cosmic microwave background. However, from the dynamical point of view, the $K = +1$ cases are even more interesting [38], and even spatially flat universes containing Yank-Mills fields exhibit chaotic oscillations of these fields [40].

The results obtained about the Einstein frame dynamics can be mapped back to the Jordan frame. The general picture obtained in the Einstein frame is that of a phase space that is free of chaos, with the orbits of the solutions converging to attractor points or going to infinity. Depending on the form of the scalar field potential, there can also be power-law attractor solutions, which are well known to exist in several scalar-

tensor gravity theories [17]. When mapped back to the Jordan frame, the non-chaotic Einstein frame dynamics translates into non-chaotic Jordan frame dynamics; when they exist, stable (unstable) equilibrium points are mapped into stable (unstable) equilibrium points, their attraction basins are deformed but they are still present, and the boundaries between different attractor points are well defined separatrices also in the Jordan frame. The possibility of such boundaries having fractal dimension, which would be a clear signature of chaos [28], is ruled out (note that fractal basin boundaries have instead been found in $K \neq 0$ FLRW, or in anisotropic universes [36]).

The dynamics of spatially flat scalar-tensor cosmologies is thus well defined and chaos-free, due to the dimensional reduction of the phase space Σ to two dimensions, as conjectured in Refs. [26, 27]. However, the proof of this conjecture is non-trivial due to the complicated structure of the phase space $\Sigma = \Sigma^+ \cup \Sigma^-$. Only spatially flat cosmologies enjoy the reduction of the phase space to two dimensions: the scale factor $a(t)$ of the FLRW metric (1.1) only appears in the combination $H \equiv \dot{a}/a$ and its first derivative \dot{H} in the field equations with $K = 0$. If $K \neq 0$ instead, terms of the form K/a^2 will appear, spoiling the reduction of the phase space to two dimensions. The surface Σ in the three-dimensional space $(H, \phi, \dot{\phi})$ separates the orbits of the solutions corresponding to $K = +1$ (located above Σ^+ or below Σ^-) from orbits corresponding to $K = -1$ (located between Σ^+ and Σ^-). This property was first realized in Ref. [37] for the special case of chaotic inflation with a massive scalar field in general relativity, and it corresponds to the impossibility of having dynamical transitions between different topologies of the spatial sections of a FLRW universe. Orbits of the solutions corresponding to $K = \pm 1$ are free to move in the three dimensions $(H, \phi, \dot{\phi})$, where there is “enough room” for them to wind around each other, so chaos can occur, and it has indeed been reported in the literature [38].

Because inflation takes the early universe extremely close to, but not exactly to, a $K = 0$ FLRW model, the orbits of the solutions can in principle depart slightly from the surface Σ , but they are extremely close to it. Then, the possibility of chaotic dynamics is not ruled out exactly, but is postponed for an extremely long time. Of course, if a second scalar field is present and significantly contributes to the dynamics of the universe, then the dimension of the phase space jumps up by two and chaos becomes possible, and is also reported in the literature (e.g., Ref. [39]) — chaos in the dynamics of two mutually coupled scalar fields is an important element of reheating after inflation.

To summarize, in addition to taking a standpoint on the possibility of chaos and generalizing the context of this debate to any $K = 0$ scalar-tensor cosmology described by the action (2.1), we have provided a general picture of the phase space and of the dynamics (see also Refs. [19]-[22] for the special case of Brans-Dicke theory, the prototype

of scalar-tensor gravity). Special choices of the arbitrary functions $\omega(\phi)$ and $V(\phi)$ will be the subject of future research.

Acknowledgments

We thank an anonymous referee for pointing out various references. This work was supported by the Natural Sciences and Engineering Research Council of Canada (*NSERC*).

References

- [1] Wald R M 1984 *General Relativity* (Chicago: Chicago University Press)
- [2] Miller A D *et al.* 1999 *Astrophys. J. Lett.* **524** L1; de Bernardis P *et al.* 2000 *Nature* **400** 955; Lange A E *et al.* 2001, *Phys. Rev. D* **63** 042001; Melchiorri A *et al.* 2000 *Astrophys. J. Lett.* **536** L63; Hanany S *et al.* 2000, *Astrophys. J. Lett.* **545**, L5; Spergel D N *et al.* 2003, *Astrophys. J. (Suppl.)* **148** 175; Bennett C L *et al.* 2003, *Astrophys. J. (Suppl.)* **148** 1; Pearson T J *et al.* 2003, *Astrophys. J.* **591** 556; Benoit A *et al.* 2003 *Astron. Astrophys.* **399** L25
- [3] Liddle A R and Lyth D H 2000 *Cosmological Inflation and Large-Scale Structure* (Cambridge: CUP)
- [4] Riess A G *et al.* 1998, *Astron. Astrophys.* **116** 1009; 1999 *Astron. J.* **118** 2668; 2001 *Astrophys. J.* **560** 49; 2004 *Astrophys. J.* **607** 665; Perlmutter S *et al.* 1998 *Nature* **391** 51; 1999 *Astrophys. J.* **517** 565; Tonry J L 2003 *et al.* *Astrophys. J.* **594** 1; Knop R *et al.* 2003 *Astrophys. J.* **598** 102; Barris B *et al.* 2004, *Astrophys. J.* **602** 571
- [5] Alcaniz J S 2004 *Phys. Rev. D* **69** 083521; Melchiorri A, Mersini L, Odman C J and Trodden M 2003, *Phys. Rev. D* **68** 043509; Hannestad S and Mortsell E 2002 *Phys. Rev. D* **66** 023510; Alam U 2004 *et al. Mon. Not. R. Astr. Soc.* **354** 275; Nesseris S and Perivolaropoulos L 2004, *Phys. Rev. D* **70**, 043531; Choudhury T R and Padmanabhan T 2005 *Astron. Astrophys.* **429** 807
- [6] Weinberg S 1989 *Rev. Mod. Phys.* **61** 1
- [7] Weinberg S 2001, in *Sources and Detection of Dark Matter*, Proceedings, Marina del Rey, California 2000, ed. by Cline D B (Berlin: Springer-Verlag)
- [8] Fang W *et al.*, hep-th/0409080; Gonzalez-Diaz P F and Jimenez-Madrid J A 2004 *Phys. Lett. B* **596** 16; Brown M G, Freese K and Kinney W H, astro-ph/0405353; Elizalde E, Nojiri S and Odintsov S D 2004 *Phys. Rev. D* **70** 043539; 2003 *Phys. Lett. B* **574** 1; hep-th/0405034; Singh P, Sami M and Dadhich N 2003 *Phys. Rev. D* **68** 023522; Hao J-G and Li X-Z 2005 *Phys. Lett. B* **606** 7; 2003 *Phys. Rev. D* **68** 043501; Li X-Z and Hao J-G 2004 *Phys. Rev. D* **69** 107303; Aguirregabiria J M, Chimento L P and Lazkoz R 2004 *Phys. Rev. D* **70** 023509; Piao Y-S and Zhang Y-Z 2004 *Phys. Rev. D* **70** 063513; Gonzalez-Diaz PF 2003 *Phys. Rev. D* **68** 021303(R); 2004 *Phys. Rev. D* **69** 063522; 2004 *Phys. Lett. B* **586** 1; Lu H Q, hep-th/0312082; Johri V B 2004 *Phys. Rev. D* **70** 041303; Hsu S D H, Jenkins A and Wise M B 2004 *Phys. Lett. B* **597** 270; Sami M and Toporensky A 2004 *Mod. Phys. Lett. A* **19** 1509; Faraoni V 2005 *Class. Quant. Grav.* **22** 3235; Buniy R V and Hsu S D H, preprint hep-th/0502203; Stefancic H 2004 *Phys. Lett. B* **586** 5; Liu D-J

- and Li X-Z 2003 *Phys. Rev. D* **68** 067301; Li X-Z and Hao J-G 2004 *Phys. Rev. D* **69** 107303; Dabrowski M P, Stachowiak T and Szydlowski M 2003 *Phys. Rev. D* **68** 103519; Babichev E, Dokuchaev V and Eroshenko Yu 2004 *Phys. Rev. Lett.* **93** 021102; Faraoni V and Israel W 2005 *Phys. Rev. D* **71** 064017; Guo Z-K, Piao Y-S and Zhang Y-Z 2004 *Phys. Lett. B* **594**, 247; Cline J M, Jeon S and Moore G D 2004 *Phys. Rev. D* **70**, 043543; Nojiri S and Odintsov S D 2003 *Phys. Lett. B* **562** 147; Mersini L, Bastero-Gil M and Kanti P 2001 *Phys. Rev. D* **64**, 043508; Bastero-Gil M, Frampton P H and Mersini L 2002 *Phys. Rev. D* **65**, 106002; Frampton P H 2003 *Phys. Lett. B* **555** 139; McInnes B, hep-th/0502209
- [9] Lucchin F, Matarrese S and Pollock M D 1986 *Phys. Lett. B* **167** 163; Salopek D S, Bond J R and Bardeen J M 1989 *Phys. Rev. D* **40** 1753; Futamase T and Maeda K I 1989 *Phys. Rev. D* **39** 399; Futamase T, Rothman T and Matzner R 1989 *Phys. Rev. D* **39** 405; Morikawa M 1989 *Astrophys. J. (Lett.)* **362** L37; 1991 *Astrophys. J.* **369** 20; Fakir R and Unruh W G 1990 *Phys. Rev. D* **41** 1783; 1990 **41** 1793; Hwang J-C 1990 *Phys. Rev. D* **42** 2601; Fakir R, Habib S and Unruh W G 1992 *Astrophys. J.* **394** 396; Fakir R and Habib S 1993, *Int. J. Mod. Phys. D* **8** 2827; Futamase T and Tanaka M 1999 *Phys. Rev. D* **60** 063511; Fakir R, gr-qc/9810054; Makino N and Sasaki M 1991 *Prog. Theor. Phys.* **86** 103; Salopek D S 1992 *Phys. Rev. Lett.* **69** 3602; Kaiser D I 1995 *Phys. Rev. D* **52** 4295; Faraoni V 1996 *Phys. Rev. D* **53** 6813; Fukuyama T *et al.* 1997 *Int. J. Mod. Phys. D* **6** 69; Hwang J-C and Noh H 1998 *Phys. Rev. Lett.* **81** 5274; 1999 *Phys. Rev. D* **60** 123001; Faraoni V and Gunzig E 1999 *Int. J. Mod. Phys. D* **8** 177; Sakai N and Yokoyama J 1999 *Phys. Lett. B* **456** 113; Chiba T 1999 *Phys. Rev. D* **60** 83508; Komatsu E and Futamase T 1998 *Phys. Rev. D* **58** 023004; 1999 **59** 064029; Faraoni V 2003 *Phys. Rev. D* **68** 063508; Szydlowski M, Czaja W and Krawiec A, astro-ph/0401293; Tsujikawa S and Gumjudpai B, *Phys. Rev. D* **69** 123523; Mukaigawa S, Muta, T and Odintsov S D 1998 *Int. J. Mod. Phys. A* **13** 2739; Faraoni V 2004 *Phys. Rev. D* **69** 123520; Carvalho F C and Saa A 2004 *Phys. Rev. D* **70** 087302; Faraoni V and Jensen M N 2006 *Class. Quant. Grav.* **23** 3005; Koh S, Kim S P and Song D J, astro-ph/0505188
- [10] Chiba T 1999 *Phys. Rev. D* **60** 083508; Uzan J-P 1999 *Phys. Rev. D* **59** 123510; Perrotta F, Baccigalupi C and Matarrese S 2000 *Phys. Rev. D* **61** 023507; Amendola L 2000 *Mon. Not. R. Astr. Soc.* **312** 521; de Ritis R, Marino A A, Rubano C and Scudellaro P 2000, *Phys. Rev. D* **62** 043506; Chen X, Scherrer R J and Steigman G 2001 *Phys. Rev. D* **63** 123504; Esposito-Far  se G, gr-qc/0011115; Sen A A and Seshadri R T 2000 *Int. J. Mod. Phys. D* **12** 445; Baccigalupi C, Matarrese S and Perrotta F 2000 *Phys. Rev. D* **62** 123510; Fujii Y 2000 *Gravit. Cosmol.* **6** 107; Fujii Y 2000 *Phys. Rev. D* **62** 044011; Bertolami O and Martins P J 2000 *Phys. Rev. D* **61** 064007; Amendola L 2000 *Phys. Rev. D* **62** 043511; 2001 *Phys. Rev. Lett.* **86** 196; Boisseau B, Esposito-Far  se G, Polarski D and Starobinsky A A 2000 *Phys. Rev. Lett.* **85** 2236; Bento M C, Bertolami O and Santos N C 2002 *Phys. Rev. D* **65** 067301; Chiba T 2001 *Phys. Rev. D* **64** 103503; Riazuelo A and Uzan J-P 2000

- Phys. Rev. D* **62** 083506; Esposito-Far  se G and Polarski D 2001 *Phys. Rev. D* **63** 063504; Banerjee N and Pavon D 2001 *Class. Quant. Grav.* **18** 593; Faraoni V 2002 *Int. J. Mod. Phys. D* **11** 471; Torres D F 2002 *Phys. Rev. D* **66** 043522; Perrotta F and Baccigalupi C, astro-ph/0205217; Riazuelo A and Uzan J-P 2002 *Phys. Rev. D* **66** 023535; Nojiri S and Odintsov S D 2003 *Phys. Rev. D* **68** 123512; hep-th/04050078; Elizalde E *et al.* 2003 *Phys. Lett. B* **574** 1; Nojiri S and Odintsov S D 2004 *Phys. Rev. D* **70** 103522; Nojiri S and Odintsov S D 2004 *Phys. Lett. B* **595** 1; Matarrese S, Baccigalupi C and Perrotta F, astro-ph/0403480; Catena R *et al.*, astro-ph/0406152
- [11] Bergmann P G 1968 *Int. J. Theor. Phys.* **1** 25; Wagoner R V 1970 *Phys. Rev. D* **1** 3209; Nordvedt K 1970 *Astrophys. J.* **161** 1059
- [12] Will C M 1993 *Theory and Experiment in Gravitational Physics* (Cambridge: CUP)
- [13] Brans C H and Dicke R H 1961 *Phys. Rev.* **124** 925
- [14] Callan C G, Friedan D, Martinec E J and Perry M J 1985 *Nucl. Phys. B* **262** 593; Fradkin E S and Tseytlin A A 1985 *Nucl. Phys. B* **261** 1
- [15] La D and Steinhardt P J 1989 *Phys. Rev. Lett.* **62** 376; Weinberg E J 1989 *Phys. Rev. D* **40** 3950; La D, Steinhardt P J and Bertschinger E 1989 *Phys. Lett. B* **231** 231; Liddle A R and Wands D 1991 *Mon. Not. Roy. Astr. Soc.* **253** 637
- [16] Accetta F S and Trester J J 1989 *Phys. Rev. D* **39** 2854; La D, Steinhardt P J and Bertschinger E 1989 *Phys. Lett. B* **231** 231; Holman R, Kolb E W and Wang Y 1990 *Phys. Rev. Lett.* **65** 17; Holman R, Kolb E W, Vadas S and Wang Y 1991 *Phys. Lett. B* **269** 252; Laycock A M and Liddle A R 1994 *Phys. Rev. D* **49** 1827; McDonald J 1993 *Phys. Rev. D* **48** 2462
- [17] Faraoni 2004 *Cosmology in Scalar-Tensor Gravity* (Dordrecht: Kluwer Academic)
- [18] Fujii Y and Maeda K 2003 *The Scalar-Tensor Theory of Gravity* (Cambridge: CUP)
- [19] Kolitch S J and Eardley D M 1995 *Ann. Phys. (NY)* **241** 128
- [20] Kolitch S J 1996 *Ann. Phys. (NY)* **246** 121
- [21] Santos C and Gregory R 1997 *Ann. Phys. (NY)* **258** 111
- [22] Holden D J and Wands D 1998 *Class. Quant. Grav.* **15** 3271
- [23] Misner C W 1969 *Phys. Rev. Lett.* **22** 1071; Khalatnikov I M and Lifschitz E M 1970 *Phys. Rev. Lett.* **24** 76; Belinskii V A, Khalatnikov I M and Lifschitz E M 1970 *Adv. Phys.* **19** 525; Barrow J D 1981 *Phys. Rev. Lett.* **46** 963; Chernoff D F and Barrow J D

- 1983 *Phys. Rev. Lett.* **50** 134; Cornish N J and Levin J 1997, *Phys. Rev. Lett.* **78** 998; 1997 *Phys. Rev. D* **55** 7489
- [24] Blanco S *et al.* 1994 *Gen. Rel. Grav.* **26** 1131
- [25] Helmi A and Vucetich H 1997 *Phys. Lett. A* **230** 153
- [26] Gunzig E *et al.* 2000 *Mod. Phys. Lett. A* **15** 1363
- [27] Gunzig E *et al.* 2000 *Class. Quant. Grav.* **17** 1783; Gunzig E *et al.* 2001 *Phys. Rev. D* **63** 067301; Saa A *et al.* 2001 *Int. J. Theor. Phys.* **40** 2295
- [28] Guckenheimer J and Holmes P 1983 *Nonlinear Oscillations, Dynamical Systems, and Bifurcation of Vector Fields* (New York: Springer); Glendinning P 1994 *Stability, Instability and Chaos: An Introduction to the Theory of Nonlinear Differential Equations* (Cambridge: CUP)
- [29] Faraoni V 2005 *Ann. Phys. (NY)* **317** 366
- [30] Foster S 1998 *Class. Quant. Grav.* **15** 3485
- [31] Amendola L, Litterio M and Occhionero F 1990 *Int. J. Mod. Phys. A* **5** 3861
- [32] Magnano G and Sokolowski L M 1994 *Phys. Rev. D* **50** 5039
- [33] Faraoni V, Gunzig E and Nardone P 1999 *Fund. Cosm. Phys.* **20** 121
- [34] Walliser D 1992 *Nucl. Phys. B* **378** 150
- [35] Faraoni V 2005 *Phys. Rev. D* **72** R061501; 2004 *Phys. Rev. D* **70** 044037; **69** 123520; 2000 *Phys. Lett. A* **269** 2009
- [36] Dettmann C P, Frankel N E and Cornish N J 1994 *Phys. Rev. D* **50** R618; 1995 *Fractals* **3** 161; Cornish N J and Levin J 1996 *Phys. Rev. D* **53** 3022; Barrow J D and Levin J 1998 *Phys. Rev. Lett.* **80** 656
- [37] Belinskii V A, Grishchuk L P, Khalatnikov I M and Zel'dovich Ya B 1985 *Phys. Lett. B* **155** 232
- [38] Page D N 1984 *Class. Quant. Grav.* **1** 417; Calzetta E and El Hasi C 1993 *Class. Quant. Grav.* **10** 1825; 1995 *Phys. Rev. D* **51** 2713; Cornish N J, Spergel D N and Starkman G D 1996 *Phys. Rev. Lett.* **77** 215; Bombelli L, Lombardo F and Castagnino M 1998 *J. Math. Phys.* **39** 6040; Cornish N J and Shellard E P S 1998 *Phys. Rev. Lett.* **81** 3571; Barrow J D 1998 *Phys. Rev. D* **57** 7204; Toporensky A V 1999 *Int. J. Mod. Phys. D* **8** 739; Kamenshchik A Yu, Khalatnikov I M, Savchenko S V and Toporensky A V 1999 *Phys. Rev. D* **59** 123516; Toporensky A V 2000, gr-qc/0011113; 2006, gr-qc/0603083;

- [39] Easter R and Maeda K 1999 *Class. Quant. Grav.* **16** 1637; Cornish N J and Levin JJ 1996; *Phys. Rev. D* **53** 3022;
- [40] Barrow J D and Levin J 1998 *Phys. Rev. Lett.* **80** 656; Jin Y and Maeda K 2005 *Phys. Rev. D* **71** 064007; Barrow J D, Jin Y and Maeda K 2005 *Phys. Rev. D* **72** 103512